Derivation of a Single-Step Hybrid Block Method with Generalized Two Off-Step Points for Solving Second Order Ordinary Differential Equation Directly.

[1]Ra'ft. Abdelrahim. and [2]Zurni. Omar

[1,2]*Department of Mathematics, College of Art and Sciences, Universiti Utara Sintok, Kedah, Malaysia*

Abstract— This paper proposes a single-step hybrid block method with generalized two off-step point for the direct solution of initial value problem of second order ordinary differential equations. The uses of power series approximate solution as an interpolation polynomial at the off points is employed in developing this method, while its second derivative is collocated at all points in the interval. Furthermore, some basic properties of the generalized method such as order, zero stability, consistency and convergence are also established. In addition, two examples of specific points of the developed method are considered to solve some initial value problems of second order ordinary differential equations. The numerical results confirm that the proposed method produces better accuracy if compared with the existing methods.

Keywords— Single-step, hybrid block method, collocation, interpolation, second order ordinary differential equations.

I. INTRODUCTION

In this article, the numerical solution to the general second order initial value problem of ordinary differential equation (ODEs) of the form

$$y'' = f\left(x, y, y'\right), \quad x \in [a, b] \tag{1}$$

with two initial conditions $y(a) = \tau_0, y'(a) = \tau_1$ is considered.

The method of reducing (1) to its equivalent system of first order has been found having some setback which includes: wastage of computer time, a lot of human effort and computational burden (see [4], [8] and [2]). Therefore, scholars have paid more attention on the establishment of direct methods for solving higher order ODEs whereby the numerical results generated are better than the method of reduction to system of first order ODEs (see [15], [11] and [3]). Some of the methods developed include the self – starting Runge - Kutta type which contains many functions to be evaluated per step ([5] and [13]) and linear multistep methods which are not self-starting but require little function to evaluate per step[12]. The implementation of implicit linear multistep method in the predictor-corrector mode is associated with a lot of human effort and computer time which renders the method to be inefficient for the use of general purpose. These weaknesses in predictor corrector method led to development of block method without predictors and do not require many functions to evaluate per step when compared with Runge- Kutta type methods, [1].

It is observed that these methods mentioned above are governed by Dahlquist barrier conditions which are extensively discussed by [6] and the introduction of hybrid methods has been used to circumvent the barrier (see [7] and [12]).The development of hybrid method with specific off step points have been considered by scholars [1], [10], [12] and [14].

In order to bring improvement in the existing method, this paper presents a single step method with generalized two off step point for solving (1) directly. This paper is divided into five sections: section two contains the derivation of the method, section three establishes the generalized basic properties of the method, section four includes specification of the method and section five includes the numerical results generated from the application of the method to second order ODE.

II. DERIVATION OF THE METHOD

Suppose the approximate solution is the power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i \tag{2}$$

where

$x \in [x_n, x_{n+1}]$ For $n = 0,1,2,\cdots,N-1,$

$a's$ are coefficient to be determined,

v is number of collocation points,

m is number of interpolation points,

$h = x_n - x_{n-1}$ is a constant step size of partition of interval $[a,b]$ which is given by

$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b..$

The second derivative of (2) is given by

$$y''(x) = f(x, y, y') = \sum_{i=0}^{v+m-1} \frac{i(i-1)a_i}{h^2}\left(\frac{x-x_n}{h}\right)^{i-2} \tag{3}$$

Interpolating (2) at x_{n+s} and x_{n+r} and collocating (3) at all points in the interval i.e x_n, x_{n+s}, x_{n+r}, x_{n+1} gives the following equations which can be written in matrix form

$$\begin{bmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 \\ 1 & r & r^2 & r^3 & r^4 & r^5 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} & \frac{20s^3}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6r}{h^2} & \frac{12r^2}{h^2} & \frac{20r^3}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_{n+s} \\ y_{n+r} \\ f_n \\ f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{bmatrix} \tag{4}$$

Gaussian elimination method is applied on (4) to find the coefficient $a's$ and then substituted into (2) to give the implicit continuous hybrid method of the form

$$y(x) = \sum_{i=s,r} \alpha_i y_{n+i} + \left(\sum_{j=0}^{1} \beta_j f_{n+j} + \sum_{i=s,r} \beta_i f_{n+i} \right) \tag{5}$$

where α_s, α_r, β_0, β_s, β_r and β_1 are given in Appendix A

Evaluating (5) at non-interpolating points i.e. x_n and x_{n+1} and evaluating its first derivative at all points gives the following equation in matrix form

$$R_0 Y_m = R_1 R^{11} + h^2 R_2 R^{22} + h^2 R_3 R^{33} \tag{6}$$

$$R_0 = \begin{bmatrix} \frac{-r}{r-s} & \frac{s}{r-s} & 0 & 0 & 0 & 0 \\ \frac{1-r}{r-s} & \frac{s-1}{r-s} & 1 & 0 & 0 & 0 \\ \frac{1}{h(r-s)} & \frac{-1}{h(r-s)} & 0 & 0 & 0 & 0 \\ \frac{1}{h(r-s)} & \frac{-1}{h(r-s)} & 0 & 1 & 0 & 0 \\ \frac{1}{h(r-s)} & \frac{-1}{h(r-s)} & 0 & 0 & 1 & 0 \\ \frac{1}{h(r-s)} & \frac{-1}{h(r-s)} & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Y_m = \begin{bmatrix} y_{n+s} \\ y_{n+r} \\ y_{n+1} \\ y'_{n+s} \\ y'_{n+r} \\ y'_{n+1} \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R^{11} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & R_2^{16} \\ 0 & 0 & 0 & 0 & 0 & R_2^{26} \\ 0 & 0 & 0 & 0 & 0 & R_2^{36} \\ 0 & 0 & 0 & 0 & 0 & R_2^{46} \\ 0 & 0 & 0 & 0 & 0 & R_2^{56} \\ 0 & 0 & 0 & 0 & 0 & R_2^{66} \end{bmatrix}, \quad R^{22} = \begin{bmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 & 0 & 0 & R_3^{14} & R_3^{15} & R_3^{16} \\ 0 & 0 & 0 & R_3^{24} & R_3^{25} & R_3^{26} \\ 0 & 0 & 0 & R_3^{34} & R_3^{35} & R_3^{36} \\ 0 & 0 & 0 & R_3^{44} & R_3^{45} & R_3^{46} \\ 0 & 0 & 0 & R_3^{54} & R_3^{55} & R_3^{56} \\ 0 & 0 & 0 & R_3^{64} & R_3^{65} & R_3^{66} \end{bmatrix}, \quad R^{33} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{bmatrix}$$

and the elements of R_2 and R_3 are given as below

$$R_3^{14} = \frac{r\left(2r^3 + 2r^2s - 5r^2 + 2rs^2 - 5rs - 3s^3 + 5s^2\right)}{60(r-s)(s-1)}$$

$$R_3^{15} = \frac{s\left(3r^3 - 2r^2s - 5r^2 - 2rs^2 + 5rs - 2s^3 + 5s^2\right)}{60(r-s)(s-1)}$$

$$R_3^{16} = \frac{-rs(2r-s)(r+s)(r-2s)}{60(r-1)(s-1)}$$

$$R_3^{34} = \frac{-\left(2r^4 + 2r^3s - 5r^3 + 2r^2s^2 - 5r^2s + 2rs^3 - 5rs^2 - 3s^4 + 5s^3\right)}{60hs(r-s)(s-1)}$$

$$R_3^{35} = \frac{-\left(3r^4 - 2r^3s - 5r^3 - 2r^2s^2 + 5r^2s - 2rs^3 + 5rs^2 - 2s^4 + 5s^3\right)}{60hr(r-s)(s-1)}$$

$$R_3^{36} = \frac{\left(2r^4 - 3r^3s - 3r^2s^2 - 3rs^3 + 2s^4\right)}{60h(r-1)(s-1)}$$

$$R_3^{45} = \frac{-(r-s)\left(3r^2 + 4rs - 5r + 3s^2 - 5s\right)}{60hr(r-1)}$$

$$R_3^{46} = \frac{(2r+3s)(r-s)^3}{60h(r-1)(s-1)}$$

$$R_3^{54} = \frac{(r-s)\left(3r^2 + 4rs - 5r + 3s^2 - 5s\right)}{60hs(s-1)}$$

$$R_3^{55} = \frac{(r-s)\left(12r^2 + 6rs - 15r + 2s^2 - 5s\right)}{60hr(r-1)}$$

$$R_3^{56} = \frac{-(3r+2s)(r-s)^3}{60h(r-1)(s-1)}$$

$$R_3^{64}=\frac{-\left(2r^4+2r^3s-5r^3+2r^2s^2-5r^2s+2rs^3+10r-3s^4+5s^3-5\right)}{60hs\left(r-s\right)\left(s-1\right)}$$

$$R_3^{65}=\frac{-3r^4+2r^3s+5r^3+2r^2s^2-5r^2s+2rs^3-5rs^2+2s^4-5s^3+10s-5}{60hr\left(r-s\right)\left(s-1\right)}$$

$$R_3^{66}=\frac{\left(2r^4-3r^3s-3r^2s^2-3rs^3+30rs-20r+2s^4-20s+15\right)}{60h\left(r-1\right)\left(s-1\right)}$$

$$R_2^{16}=\frac{r\left(2r^3-3r^2s-5r^2-3rs^2+15rs+2s^3-5s^2\right)}{60}$$

$$R_2^{26}=\frac{\left(r-1\right)\left(s-1\right)\left(r+s-2\right)\left(r-2s+1\right)\left(2r-s-1\right)}{60rs}$$

$$R_2^{36}=\frac{-\left(2r^4-3r^3s-5r^3-3r^2s^2+15r^2s-3rs^3+15rs^2+2s^4-5s^3\right)}{60hrs}$$

$$R_2^{46}=\frac{-\left(r-s\right)^3\left(2r+3s-5\right)}{60hrs}$$

$$R_2^{56}=\frac{\left(r-s\right)^3\left(3r+2s-5\right)}{60hrs}$$

$$R_2^{66}=\frac{3r^3s-2r^4+5r^3+3r^2s^2-15r^2s+3rs^3-15rs^2+30rs-10r-2s^4+5s^3-10s+5}{60hrs}$$

Multiplying Equation (6) by R_0^{-1} gives the hybrid block method

$$R_0Y_m=R_1R^{11}+h^2R_2R^{22}+h^2R_3R^{33} \qquad (8)$$

$$R^0=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\quad R^1=\begin{bmatrix}0&0&1&0&0&sh\\0&0&1&0&0&rh\\0&0&1&0&0&h\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}$$

$$R^2=\begin{bmatrix}0&0&0&0&0&\frac{-s^2\left(20s-5r-5rs+2r^2\right)}{60r}\\0&0&0&0&0&\frac{r^2\left(20s-5r-5rs+2r^2\right)}{60s}\\0&0&0&0&0&\frac{\left(20rs-5s-5r+2\right)}{60rs}\\0&0&0&0&0&\frac{-s\left(2s-6r+2sr-s^2\right)}{12hr}\\0&0&0&0&0&\frac{r\left(6s-2r+2sr+r^2\right)}{12hs}\\0&0&0&0&0&\frac{\left(6rs-2s-2r+1\right)}{12hrs}\end{bmatrix} \text{ and}$$

$$R^3=\begin{bmatrix}0&0&0&\frac{s^2\left(5s-10r+5rs-3s^2\right)}{60\left(r-1\right)\left(s-1\right)}&\frac{s^4\left(2s-5\right)}{60r\left(r-1\right)\left(r-s\right)}&\frac{s^4\left(5r-2s\right)}{60\left(r-1\right)\left(s-1\right)}\\0&0&0&\frac{s^4\left(2s-5\right)}{60s\left(r-1\right)\left(r-s\right)}&\frac{r^2\left(10s-5r+5rs-3r^2\right)}{60\left(r-s\right)\left(r-1\right)}&\frac{s^4\left(2r-5s\right)}{60\left(r-1\right)\left(s-1\right)}\\0&0&0&\frac{-\left(5r-5\right)}{60s\left(s-1\right)\left(r-s\right)}&\frac{\left(5s-2\right)}{60r\left(r-1\right)\left(r-s\right)}&\frac{\left(10rs-5s-5r+3\right)}{60\left(s-1\right)\left(r-1\right)}\\0&0&0&\frac{s\left(4s-6r+4rs-3s^2\right)}{12h\left(r-s\right)\left(s-1\right)}&\frac{-s^3\left(s-2\right)}{12rh\left(r-1\right)\left(r-s\right)}&\frac{s^3\left(2r-s\right)}{12h\left(r-1\right)\left(s-1\right)}\\0&0&0&\frac{s^3\left(r-2\right)}{12sh\left(s-1\right)\left(r-s\right)}&\frac{-r\left(6s-4r-4rs+3r^2\right)}{12h\left(r-1\right)\left(r-s\right)}&\frac{-r^3\left(r-2s\right)}{12h\left(r-1\right)\left(s-1\right)}\\0&0&0&\frac{-\left(2r-1\right)}{12sh\left(r-s\right)\left(s-1\right)}&\frac{s\left(2s-1\right)}{12rh\left(r-s\right)\left(s-1\right)}&\frac{s\left(6rs-4s-4r+3\right)}{12h\left(r-1\right)\left(s-1\right)}\end{bmatrix}$$

Equation (8) can also written as

$$y_{n+s}=y_n+hsy'_n-\frac{h^2s^2\left(5s-20rs-2s^2\right)}{60r}f_n-\frac{h^2s^4\left(2s-5\right)}{60r\left(r-s\right)\left(r-1\right)}f_{n+r}+\frac{h^2s^2\left(5s-10r-5rs-3s^2\right)}{60\left(r-s\right)\left(s-1\right)}f_{n+s}+\frac{h^2s^4\left(5r-2s\right)}{60\left(r-1\right)\left(s-1\right)}f_{n+1}$$

$$y_{n+r}=y_n+hry'_n+\frac{h^2r^2\left(20s-5r-5rs+2r^2\right)}{60s}f_n+\frac{h^2r^4\left(2r-5\right)}{60s\left(r-s\right)\left(s-1\right)}f_{n+s}+\frac{h^2r^2\left(10s-5rs+3r^2\right)}{60\left(r-s\right)\left(r-1\right)}f_{n+r}-\frac{h^2r^4\left(2r-5s\right)}{60\left(r-1\right)\left(s-1\right)}f_{n+1}$$

$$y_{n+1}=y_n+hy'_n+\frac{h^2\left(20rs-5s-5r+2\right)}{60sr}f_n-\frac{h^2\left(5r-2\right)}{60s\left(r-s\right)\left(s-1\right)}f_{n+s}+\frac{h^2\left(5s-2\right)}{60r\left(r-s\right)\left(r-1\right)}f_{n+r}-\frac{h^2\left(10sr-5s-5r+3\right)}{60\left(r-1\right)\left(s-1\right)}f_{n+1}$$

$$y'_{n+s}=y'_n-\frac{hs\left(2s-6r+2rs-s^2\right)}{12r}f_n+\frac{hs\left(4s-6r+4rs-3s^2\right)}{12\left(r-s\right)\left(s-1\right)}f_{n+s}-\frac{hs^3\left(s-2\right)}{12r\left(r-s\right)\left(r-1\right)}f_{n+r}+\frac{hs^3\left(2r-s\right)}{12\left(r-1\right)\left(s-1\right)}f_{n+1}$$

$$
\begin{aligned}
y'_{n+r} &= y'_n + \frac{hr\left(6s-2r-2rs-r^2\right)}{12r} f_n \\
&+ \frac{hr^3(r-2)}{12s(r-s)(s-1)} f_{n+s} + \frac{hr\left(6s-4r-4rs+3r^2\right)}{12(r-s)(r-1)} f_{n+r} \\
&- \frac{hr^3(r-2s)}{12(r-1)(s-1)} f_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+1} &= y'_n + \frac{h(6rs-2s-2r+1)}{12rs} f_n - \frac{hr^3(2r-1)}{12s(r-s)(s-1)} f_{n+s} \\
&+ \frac{h(2s-1)}{12r(r-s)(r-1)} f_{n+r} - \frac{h(6rs-4s-4r+3)}{12(r-1)(s-1)} f_{n+1}
\end{aligned}
$$

III. PROPERTIES OF THE METHOD

A. Order of the Method

Definition 1: The linear difference operator L associated with (7) is defined as

$$L\left[y(x);h\right] = R^0 Y_m - R^1 R^{11} - h^2 R^2 R^{22} - h^2 R^3 R^{33} \qquad (9)$$

where *y(x)* is an arbitrary test function continuously differentiable on $\left[a,b\right]$.

Expanding Y_m and R^{33}component respectively in Taylor's series and collecting terms in powers of *h* gives

$$
\begin{aligned}
L\left[y(x);h\right] &= \overline{\overline{C}}_0 y(x) + \overline{\overline{C}}_1 h y'(x) \\
&+ \overline{\overline{C}}_2 h^2 y''(x) + \cdots + \overline{\overline{C}}_p h^p y^p(x) + \cdots
\end{aligned} \qquad (10)
$$

where $\bar{\bar{c}}_j,\ j = 0,1,\cdots$ are vectors

Definition 2: The hybrid block method (7) and the associated linear difference operator (9) are said to have order p if $\overline{\overline{C}}_0 = \overline{\overline{C}}_1 = \overline{\overline{C}}_2 = \cdots = \overline{\overline{C}}_p = \overline{\overline{C}}_{p+1} = \overline{\overline{0}}$ and $\overline{\overline{C}}_{p+2} \neq 0$. $\overline{\overline{C}}_{p+2}$ is called vector of error constant.

Expand (9) in Taylor series about x_n gives

$$
\left[
\begin{array}{l}
\sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!} y_n^j + \frac{h^2 s^2(5s-20rs-2s^2)}{(60r)} y''_n - \frac{s^2\left(5s-10r-5rs-3s^2\right)}{60(r-s)(s-1)} \sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!} y_n^{j+2} \\
+\frac{s^4(2s-5)}{60r(r-s)(r-1)} \sum_{j=0}^{\infty}\frac{(r)^j h^{j+2}}{j!} y_n^{j+2} - \frac{s^4(5r-2s)}{60(r-1)(s-1)} \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} y_n^{j+2} - sh\ y'_n - y_n \\
\sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!} y_n^j - \frac{h^2 r^2\left(20s-5r-5rs+2r^2\right)}{60s} y'' - \frac{r^4(2r-5)}{60s(r-s)(s-1)} \sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!} y_n^{j+2}{}_n \\
-\frac{r^2\left(10s-5rs+3r^2\right)}{60(r-s)(r-1)} \sum_{j=0}^{\infty}\frac{(r)^{j+2} h^{j+2}}{j!} y_n^{j+2} + \frac{r^4(2r-5s)}{60(r-1)(s-1)} \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} y_n^{j+2} - rh\ y'_n - y_n \\
\sum_{j=0}^{\infty}\frac{h^j}{j!} y_n^j - y_n - \frac{h^2(20rs-5s-5r+2)}{60sr} y''_n - \frac{(5r-2)}{60s(r-s)(s-1)} \sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!} y_n^{j+2} \\
-\frac{(5s-2)}{60r(r-s)(r-1)} \sum_{j=0}^{\infty}\frac{(r)^j h^{j+2}}{j!} y_n^{j+2} + \frac{(10sr-5s-5r+3)}{60(r-1)(s-1)} \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} y_n^{j+2} - h\ y'_n \\
\sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!} y_n^{j+1} - y'_n + \frac{hs\left(2s-6r+2rs-s^2\right)}{12r} y''_n - \frac{s^3(2r-s)}{12(r-1)(s-1)} \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} y_n^{j+2} \\
+\frac{s^3(s-2)}{12r(r-s)(r-1)} \sum_{j=0}^{\infty}\frac{(r)^j h^{j+1}}{j!} y_n^{j+2} - \frac{s\left(4s-6r+4rs-3s^2\right)}{12(r-s)(s-1)} \sum_{j=0}^{\infty}\frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!} y_n^{j+1} - y'_n - \frac{hr\left(6s-2r-2rs-r^2\right)}{12r} y''_n + \frac{1}{144} \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} y_n^{j+2} \\
-\frac{r\left(6s-4r-4rs+3r^2\right)}{12(r-s)(r-1)} \sum_{j=0}^{\infty}\frac{(r)^j h^{j+1}}{j!} y_n^{j+2} - \frac{r^3(r-2)}{12s(r-s)(s-1)} \sum_{j=0}^{\infty}\frac{(s)^j h^{j+1}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty}\frac{h^j}{j!} y_n^{j+1} - y'_n - \frac{h(6rs-2s-2r+1)}{12rs} y''_n + \frac{(6rs-4s-4r+3)}{12(r-1)(s-1)} \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} y_n^{j+2} \\
-\frac{(2s-1)}{12r(r-s)(r-1)} \sum_{j=0}^{\infty}\frac{(r)^j h^{j+1}}{j!} y_n^{j+2} + \frac{r^3(2r-1)}{12s(r-s)(s-1)} \sum_{j=0}^{\infty}\frac{(s)^j h^{j+1}}{j!} y_n^{j+2}
\end{array}
\right]
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Comparing the coefficient of *h* led to $\overline{\overline{C}}_0 = \overline{\overline{C}}_1 = \overline{\overline{C}}_2 = \overline{\overline{C}}_3 = \overline{\overline{C}}_4 = \overline{\overline{C}}_5 = \left[0,0,0,0,0,0\right]^T$

Therefore, the order of the method is $[4,4,4,4,4,4]^T$ for all

$$
s, r \in (0,1) \setminus \left\{
\begin{array}{l}
\left\{ r = \dfrac{s^2-2s}{-5+2s} \right\} \cup \left\{ s = \dfrac{2r+r^2}{5-2r} \right\} \cup \left\{ s = \dfrac{1-2r}{2-5r} \right\} \\
\cup \left\{ r = \dfrac{3s^2-5s}{5s-10} \right\} \cup \left\{ s = \dfrac{3r^2-5r}{5r-10} \right\} \cup \left\{ r = \dfrac{5s-3}{10s-5} \right\}
\end{array}
\right\}
$$

with general error constants vector

$$\overline{\overline{C}}_6 = \begin{bmatrix} \dfrac{s^4\left(2s-5r+2rs-s^2\right)}{1440} \\ \dfrac{-r^4\left(5s-2r-2rs-r^2\right)}{1440} \\ \dfrac{-\left(5rs-2s-2r+1\right)}{1440} \\ \dfrac{s^3\left(5s-10r+5rs-3s^2\right)}{1440} \\ \dfrac{r^3\left(5r-10s+5rs-3r^2\right)}{1440} \\ \dfrac{-\left(10sr-5s-5r+3\right)}{1440} \end{bmatrix}$$

B. Zero stability of the method

Definition 4: The hybrid block method (7) is said to be zero stable if the first characteristic polynomial $\pi(z)$ having roots such that $|z_r| \le 1$ and if $|z_r| = 1$ then the multiplicity of z_r must not greater than two.

$$\pi(z) = \left|zR^0 - R^1\right|$$

$$= \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & sh \\ 0 & 0 & 1 & 0 & 0 & rh \\ 0 & 0 & 1 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right|$$

$$= \quad z^4(z-1)^2$$

whose solution is $z = 0,0,0,0,1,1$. Hence, our method is zero stable for all $s, r \in (0,1)$

C. *Consistency*

Definition 3: The one step hybrid block method (7) is said to be consistent if order of the method greater than or equal one i.e. $P \ge 1$

Since in our new method $\overline{\overline{C}}_6 = \overline{\overline{C}}_{4+2} \ne 0,$, this implies $P = 4$. Hence, our method is consistent by definition 3.

D. Convergence

Theorem1. [9]: Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent

Therefore, since the new hybrid block method is consistence and zero stabile, it can be concluded that the method is convergent.

F. Region of Absolute Stability

In this article, the locus method was adopted to determine the region of absolute stability. The method (7) is said to be absolutely stable if for a given h all roots of the characteristic polynomial $\pi(z,h) = \rho(z) - \overline{h}\sigma(z)$, satieties $|z_t| < 1$. The test equation $y'' = -\lambda^2 y$ is substituted in (7) where $\overline{h} = \lambda^2 h^2$ and $\lambda = \dfrac{df}{dy}$. Substituting $z = \cos\theta - i\sin\theta$ and considering real part yields

$$\overline{h}(\theta) = \frac{\left(1440\cos\theta - 1440\right)}{\left(2r^2s^2 + 4rs - 3rs^2 - 3r^2s + r^2s^2\cos\theta\right)} \tag{12}$$

IV. SPECIFICATION

This section considers two specific numerical methods of two hybrid points.

Method A

Substituting $s = \dfrac{1}{10}$, $r = \dfrac{1}{5}$ into equation (8), the following block of one step with two hybrid points and its derivative are obtained

$$y_{n+\frac{1}{10}} = y_n + h\frac{1}{10}y'_n + \frac{57h^2}{20000} f_n + \frac{h^2}{540000} f_{n+1} - \frac{h^2}{2000} f_{n+\frac{2}{10}} + \frac{143h^2}{54000} f_{n+\frac{1}{10}}$$

$$y_{n+\frac{2}{10}} = y_n + \frac{1}{5}h\,y'_n + \frac{49h^2}{7500} f_n + \frac{h^2}{270000} f_{n+1} - \frac{h^2}{6000} f_{n+\frac{1}{5}} + \frac{46h^2}{3375} f_{n+\frac{1}{10}}$$

$$y_{n+1} = y_n + hy'_n + \frac{3h^2}{4} f_n + \frac{17h^2}{432} f_{n+1} + \frac{25h^2}{16} f_{n+\frac{1}{5}} - \frac{50h^2}{27} f_{n+\frac{1}{10}}$$

$$y'_{n+\frac{1}{10}} = y'_n + \frac{27h}{2400} f_n + \frac{h}{28800} f_{n+1} - \frac{19h}{1920} f_{n+\frac{1}{5}} + \frac{5h}{72} f_{n+\frac{1}{10}}$$

$$y'_{n+\frac{2}{10}} = y'_n - \frac{h}{30} f_n + \frac{h}{30} f_{n+\frac{1}{5}} + \frac{2h}{15} f_{n+\frac{1}{10}}$$

$$y'_{n+1} = y'_n + \frac{13h}{6} f_n + \frac{2h}{9} f_{n+1} + \frac{25h}{6} f_{n+\frac{1}{5}} - \frac{50h}{9} f_{n+\frac{1}{10}}$$

Replacing $s=\frac{1}{10}$, $r=\frac{1}{5}$ into Equation(11) gives the order of the method A to be $\left[4,4,4,4,4,4\right]^T$ with error constant

$$\overline{\overline{C}}_6=\begin{bmatrix}-5.347222e^{-8}\\-2.222222e^{-8}\\-3.472222e^{-4}\\-9.930556e^{-7}\\-1.111111e^{-7}\\-1.180556e^{-3}\end{bmatrix}$$

After substituting the values of s and r in (12), we get the stability interval of (-46154, 0) as shown in Figure 1.

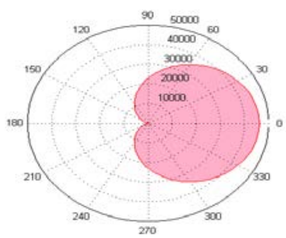

Fig. 1 Stability Region of single step hybrid block method A

Method B

$$y_{n+\frac{1}{3}}=y_n+\frac{1}{3}hy'_n+\frac{433h^2}{14580}f_n+\frac{37h^2}{9720}f_{n+1}-\frac{208h^2}{18225}f_{n+\frac{3}{4}}+\frac{59h^2}{1800}f_{n+\frac{1}{3}}$$

$$y_{n+\frac{3}{4}}=y_n+\frac{3}{4}hy'_n+\frac{201h^2}{2560}f_n+\frac{27h^2}{5120}f_{n+1}-\frac{3h^2}{1600}f_{n+\frac{3}{4}}+\frac{5103h^2}{25600}f_{n+\frac{1}{3}}$$

$$y_{n+1}=y_n+hy'_n+\frac{19h^2}{180}f_n+\frac{h^2}{120}f_{n+1}+\frac{16h^2}{225}f_{n+\frac{3}{4}}+\frac{63h^2}{200}f_{n+\frac{1}{3}}$$

$$y'_{n+\frac{1}{3}}=y'_n+\frac{31h}{243}f_n+\frac{7h}{324}f_{n+1}-\frac{16h}{243}f_{n+\frac{3}{4}}+\frac{1h}{4}f_{n+\frac{1}{3}}$$

$$y'_{n+\frac{3}{4}}=y'_n+\frac{27h}{256}f_n-\frac{9h}{512}f_{n+1}+\frac{3h}{16}f_{n+\frac{3}{4}}+\frac{243h}{512}f_{n+\frac{1}{3}}$$

$$y'_{n+1}=y'_n+\frac{h}{9}f_n+\frac{h}{12}f_{n+1}+\frac{16h}{45}f_{n+\frac{3}{4}}+\frac{9h}{20}f_{n+\frac{1}{3}}$$

Using the same procedure as previously described, the order of the block method B is $\left[4,4,4,4,4,4\right]^T$ with error constant

$$\overline{\overline{C}}_6=\begin{bmatrix}-2.310052e^{-5}\\1.968384e^{-4}\\-5.787037e^{-5}\\-1.264575e^{-4}\\-6.103516e^{-6}\\-5.787037e^{-5}\end{bmatrix}$$

Replacing $s=\frac{1}{3}$, $r=\frac{3}{4}$ in (12) gives the stability interval of (-11520, 0). Refer to Figure 2.

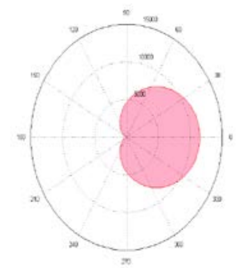

Fig. 2 stability Region of single step hybrid block method B

V. NUMERICAL EXPERIMENT

The performance of the two specific hybrid block methods(method A and method B) is tasted on the following two second initial value problems

Problem 1:

$$y''-x\left(y'\right)^2=0,\ y\left(0\right)=1,\ y'\left(0\right)=\frac{1}{2}.$$

Exact solution: $y=1+\frac{1}{2}ln\left(\frac{2+x}{2-x}\right)$ with $h=\frac{1}{320}$.

Table 1 Exact solution and computed solution of the new method A for solving Problem 1

x	Exact solution	Computed solution in A
0.1	1.050041729278491400	1.050041729278492000
0.2	1.100335347731075300	1.100335347731076000
0.3	1.151140435936466500	1.151140435936468100
0.4	1.202732554054081600	1.202732554054083400
0.5	1.255412811882994600	1.255412811882994100
0.6	1.309519604203111900	1.309519604203104500
0.7	1.365443754271397100	1.365443754271378400
0.8	1.423648930193603500	1.423648930193562400
0.9	1.484700278594054600	1.484700278593966500
1.0	1.549306144334058600	1.549306144333876900

Table 2 Exact solution and computed solution of the new method B for solving Problem 1

x	Exact solution	Computed solution
0.1	1.050041729278491400	1.05004172927849030
0.2	1.100335347731075300	1.10033534773107400
0.3	1.151140435936466500	1.15114043593646390
0.4	1.202732554054081600	1.20273255405407540
0.5	1.255412811882994600	1.25541281188297900
0.6	1.309519604203111900	1.30951960420307810
0.7	1.365443754271397100	1.36544375427133140
0.8	1.423648930193603500	1.42364893019348430
0.9	1.484700278594054600	1.48470027859383440
1.0	1.549306144334058600	1.54930614433365490

Table 3 Comparison of the new methods with [2] for solving Problem 1

x	ERROR IN A	Error in [2]	Error in B
0.1	$2.2204e^{-16}$	$2.5056e^{-12}$	$1.11022e^{-15}$
0.2	$6.6613e^{-16}$	$2.0446e^{-11}$	$1.33226e^{-15}$
0.3	$1.5543e^{-15}$	$7.0966e^{-11}$	$2.66453e^{-15}$
0.4	$1.7763e^{-15}$	$1.7482e^{-10}$	$6.21724e^{-15}$
0.5	$4.4408e^{-16}$	$3.5904e^{-10}$	$1.55431e^{-14}$
0.6	$7.3274e^{-15}$	$6.6068e^{-10}$	$3.37507e^{-14}$
0.7	$1.8651e^{-14}$	$1.1328e^{-9}$	$6.57252e^{-14}$
0.8	$4.1078e^{-14}$	$1.8543e^{-9}$	$1.19238e^{-13}$
0.9	$8.8151e^{-14}$	$2.9461e^{-9}$	$2.20268e^{-13}$
1.0	$1.8163e^{-13}$	$4.6013e^{-9}$	$4.03677e^{-13}$

Problem 2:

$$y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0,\ y(1) = 1,\ y'(1) = 1.$$

Exact solution: $y = \frac{5}{3x} - \frac{2}{3x^4}$, with $h = \frac{1}{320}$.

Table 4 Exact solution and computed solution of the new method A for solving Problem 2

x	Exact solution	Computed solution A
1.0094	1.0089449950888376	1.0089449950886735
1.0125	1.0117410181679887	1.0117410181676894
1.0156	1.0144475426864139	1.0144475426859429
1.0188	1.0170664942356729	1.0170664942349952
1.0219	1.0195997547562881	1.0195997547553703
1.0250	1.0220491636294322	1.0220491636282434
1.0281	1.0244165187384029	1.0244165187369134
1.0313	1.0267035775008062	1.0271515055533533
1.0094	1.0089449950888376	1.0089449950886735
1.0125	1.0117410181679887	1.0117410181676894

Table 5 Exact solution and computed solution of the new method B for solving Problem 2

x	Exact solution	Computed solution in B
1.0094	1.0089449950888376	1.0089449950888227
1.0125	1.0117410181679887	1.0117410181679631
1.0156	1.0144475426864139	1.0144475426863755
1.0188	1.0170664942356729	1.0170664942356189
1.0219	1.0195997547562880	1.0195997547562161
1.0250	1.0220491636294322	1.0220491636293405
1.0281	1.0244165187384029	1.0244165187382896
1.0313	1.0267035775008062	1.0267035775006690
1.0094	1.0089449950888376	1.0089449950888227
1.0125	1.0117410181679887	1.0117410181679631

Table 6 Comparison of the new methods with [2] for solving Problem 2

x	ERROR IN A	Error in[2]	Error in B
1.0094	$1.6409e^{-13}$	$2.0169e^{-10}$	$1.4876e^{-14}$
1.0125	$2.9931e^{-13}$	$4.5540e^{-10}$	$2.5535e^{-14}$
1.0156	$4.7095e^{-13}$	$7.9967e^{-10}$	$3.8413e^{-14}$
1.0188	$6.7768e^{-13}$	$1.2305e^{-9}$	$5.3956e^{-14}$
1.0219	$9.1771e^{-13}$	$1.7440e^{-9}$	$7.1942e^{-14}$
1.0250	$1.1888e^{-12}$	$2.3365e^{-9}$	$9.1704e^{-14}$
1.0281	$1.4894e^{-12}$	$3.0043e^{-9}$	$1.1324e^{-13}$
1.0313	$1.8887e^{-12}$	$3.7441e^{-9}$	$1.3722e^{-13}$

VI. COONCLUSION

This paper has successfully developed a new single-step hybrid block method with generalized two off-step points for solving second ODEs. The zero stability, consistency, convergence, order, region of absolute stability and error constant of the developed method are also examined. The proposed method not only possesses good properties of a numerical method, it has also been proven to be superior than the existing methods in term of accuracy when solving the same initial value problems of second order ODEs directly. Hence, this method should be considered as a viable alternative for solving initial value problem of second order ODEs. Furthermore, the developed method can be extended to solve a system of initial value problem of higher ODEs directly.

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Appendix A

$$\alpha_r = \frac{-(x_n - x + hs)}{h(r-s)}$$
$$\alpha_s = \frac{(x_n - x + hr)}{h(r-s)}$$

$$\beta_0 = \frac{((x_n - x + hs)(x_n - x + hr)}{(60rsh^3)}(2h^3r^3 - 3h^3r^2s - 5h^3r^2 - 3h^3rs^2 + 15h^3rs + 3x_n^3$$
$$-5h^3s^2 + 2h^2r^2x - 2h^2r^2x_n - 3h^2rsx + 3h^2rsx_n - 5h^2rx + 5h^2rx_n + 2h^2s^2x + 5hx^2$$
$$-5h^2sx + 5h^2sx_n + 2hrx_n^2 + 2hsx^2 - 4hsxx_n + 2hsx_n^2 + 5hx_n^2 - 3x^3 + 2h^3s^3 + 2hrx^2$$
$$-4hrxx_n + 9x^2x_n - 9xx_n^2 - 2h^2s^2x_n - 10hxx_n)$$

$$\beta_s = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^3s(s-1)(r-s))}(2h^3r^3 + 2h^3r^2s - 5h^3r^2 + 2h^3rs^2 - 5h^3rs - 3h^3s^3$$
$$+3x_n^3 + 5h^3s^2 + 2h^2r^2x - 2h^2r^2x_n + 2h^2rsx - 2h^2rsx_n - 5h^2rx + 5h^2rx_n - 3h^2s^2x$$
$$-3x^3 + 5h^2sx + 2hrx^2 - 4hrxx_n + 2hrx_n^2 - 3hsx^2 + 6hsxx_n - 3hsx_n^2 + 5hx^2 + 9x^2x_n$$
$$-5h^2sx_n - 10hxx_n + 3h^2s^2x_n + 5hx_n^2 - 9xx_n^2)$$

$$\beta_r = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^3r(r-1)(r-s))}(3h^3r^3 - 2h^3r^2s - 5h^3r^2 - 2h^3rs^2 + 5h^3rs - 2h^3s^3$$
$$+5h^3s^2 + 3h^2r^2x - 3h^2r^2x_n - 2h^2rsx + 2h^2rsx_n - 5h^2rx + 5h^2rx_n - 2h^2s^2x + 3x^3$$
$$+5h^2sx - 5h^2sx_n + 3hrx^2 - 6hrxx_n + 3hrx_n^2 - 2hsx^2 + 4hsxx_n - 2hsx_n^2 - 5hx^2$$
$$-5hx_n^2 + 10hxx_n + 2h^2s^2x_n - 9x^2x_n + 9xx_n^2 - 3x_n^3)$$

$$\beta_1 = -\frac{(x_n - x + hs)(x_n - x + hr)}{(60h^3(s-1)(r-1))}(2h^3r^3 - 3h^3r^2s - 3h^3rs^2 + 2h^3s^3 + 2h^2r^2x$$
$$-2h^2r^2x_n - 3h^2rsx + 3h^2rsx_n + 2h^2s^2x - 2h^2s^2x_n + 2hrx^2 - 4hrxx_n + 2hrx_n^2$$
$$-4hsxx_n + 2hsx_n^2 + 2hsx^2 + 9x^2x_n - 9xx_n^2 + 3x_n^3 - 3x^3)$$